ON CONSTRUCTION OF SYMMETRICAL FRACTIONAL FACTORIAL DESIGNS

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1. Introduction

Das (1964) developed an alternative approach for construction of symmetrical factorial designs and obtaining maximum number of factors. The same approach has been extended in this paper for the construction of symmetrical fractional factorial designs with confounding. This method of construction automatically solves the problem of selecting the factors that are to be suppressed in analysing the fractional factorial designs. The converse problem, *viz.*, given the treatment combinations in a fractional factorial design finding out the identity contrasts and confounded interactions has also been solved through this approach in a simpler way.

In the last section, the method of obtaining the contents of the key block of a symmetrical factorial design given a set of confounded interactions, through the method of Das (1964), has also been described.

2. The Method

Let there be n factors each at 2 levels and let the 2 elements, 0 and 1, be the 2 levels of each of the factors.

For the construction of the fractional design $1/2^m$ $[2^n]$, we shall follow the method given by Das (1964) and at first we take the $(n-m)$ independent treatment combinations of $(n-m)$ factors each at 2 levels denoted by 0 and 1 (vide Table on next page).

The $(n-m \times n-m)$ square formed of the $(n-m)$ independent treatment combinations of $(n-m)$ factors can be extended by introducing m further columns such that each column consists of

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$(n-m)$ elements, 0 or 1 to form a scheme of $(n-m)$ rows and n columns ensuring that each of the $(n-r)$ columns contain at least one

Basic Factors

		A_1	A_2	A_3	...	...	...	A_{n-m}
	1	1	0	0	...	...	...	0
Independent	2	0	1	0	...	...	...	0
treatment	3	0	0	1	...	...	...	0
combinations.	...	...	...	...	...	...	...	...
	...	...	...	...	...	...	...	...
	$(n-m)$	0	0	0	...	...	...	1

non-zero element which would keep the main effect of the factor free from block effect. Writing the factor notation A_{n-m+1}, $A_{n-m+2}, \ldots, A_n$ (Added Factors) above these m columns subsequently introduced, the rows of this scheme give $(n-m)$ independent treatment combinations of n factors each at 2 levels and from these combinations a total of $2^{n-m}-1$ treatment combinations of n factors can be obtained as usual. These with the control treatment give the $1/2^m$th replicate of 2^n design. This design is actually the key block of size 2^{n-m} of a fully replicated confounded design.

As described by Das (1964) we can find from the m independent added columns, m independent interactions which are confounded in the fully replicated design. In the fractional design, these interactions constitute the identity group of interactions along with their generalised interactions.

The next problem is to arrange the 2^{n-m} treatments in blocks of size, say 2^r. For this we first obtain the key block from which the other blocks can be obtained as usual. As contents of the key block we shall first obtain r independent treatment combinations in it. For this we obtain r rows in such a way that its first r columns form a $(r \times r)$ unit matrix. These r rows have to be obtained from the $(n-m)$ rows each of n elements obtained above for getting the fractional design, by taking some rows out of the $(n-m)$ rows as they are if suitable, or by forming suitable rows by adding two or more of them. The last $(n-r)$ columns of the above r rows give $(n-r)$ independent interactions which are confounded together with those that form the identity group of interactions. From these interactions $2^{n-r}-1$ interactions can be generated and the set of interactions contain all the interactions that give the identity group as also all the alias sub-group of confounded interactions.

The next problem is to separate the identity group of interactions from the rest for getting the interactions confounded. For this purpose we first take all the interactions in the identity group, next another interaction out of the $2^{n-r}-1$ interactions obtained above which has not occurred in the identity group is taken. Its alias sub-group of interactions is then obtained. This gives us the one alias group of confounded interactions. Next another interaction out of the $2^{n-r}-1$ interactions which has not occurred either in the identity group or in the alias group of interactions confounded is taken. Its alias sub-group is also obtained and this gives us the second alias group of confounded interactions. In this way, it will be possible to obtain all the confounded interactions.

The precautions given by Das (1964) to save main effects and two factor interactions are necessary here also. While analysing the above design we have to suppress all the m factors which correspond to the m columns added for obtaining the $1/2^m$th fraction of the 2^n factorial. For illustrating the above method an example is given below.

Consider $1/2^2$ $[2^8, 2^4]$ design. Here we have first to take an arrangement of 6 basic factors and 2 added factors as shown below in Scheme I.

	Basic Factors.						Added Factors	
	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
	1	0	0	0	0	0	1	1
Scheme I:	0	1	0	0	0	0	1	1
Independent	0	0	1	0	0	0	1	0
treatment	0	0	0	1	0	0	1	0
combinations	0	0	0	0	1	0	0	1
	0	0	0	0	0	1	0	1

The added columns have been so chosen that each column contains the maximum number of 1's along with their sums two by two, three by three etc. The identity contrasts obtained from the added columns are *ABCDG*, *ABEFH*, and *CDEFGH*. We can thus suppress the factors *G* and *H* in the analysis, which are added factors.

The next problem is to arrange the 2^6 treatment combinations including control treatment, which are generated by the above 6 independent treatment combinations, in blocks of size 2^4.

Scheme II :	*Basic Factors* *A*	*B*	*C*	*D*	*Added Factors* *E*	*F*	*G*	*H*	*Obtained by adding rows of Scheme I.*
Independent treatment combinations	1	0	0	0	1	0	1	0	(1) & (5)
	0	1	0	0	0	0	1	1	(2)
	0	0	1	0	0	1	1	1	(3) & (6)
	0	0	0	1	1	1	1	0	(4), (5) & (6)

The above rows of Scheme II which form a (4×4) unit matrix with its first 4 columns be obtained from Scheme I by taking some of its rows as they are or by forming fresh rows by adding the rows of Scheme I two by two, three by three, etc. If necessary, interchanging of columns is also permissible provided care is taken to see that the factors corresponding to the columns also move with them. The 4 rows above in Scheme II which give the key block of the confounded design can be obtained in many ways through the method described.

All the interactions confounded to get the key block are obtained from the following 4 independent interactions, *viz.*, *ADE*, *CDF*, *ABCDG*, *BCH*. Their generalised interactions are *ACEF*, *BCEG*, *ABCDEH*, *ABFG*, *BDFH*, *ADGH*, *BDEFG*, *ABEFH*, *ACFGH*, *EGH*, & *CDEFGH*. Now we know that the identity group is formed by *ABCDG*, *ABEFH* & *CDEFGH*. We now take *ADE* which has not occurred in the identity group. Its alias sub-group is $ADE = BCEG = BDFH = ACFGH$. Next we take the interaction *EGH*, which has not occurred in the identity group and in the above alias sub-group. Its alias sub-group is $EGH = ABCDEH = ABFG = CDF$. Next we take *BCH* and its alias sub-group is $BCH = ADGH = ACEF = BDEFG$. In this way it is possible to separate out the identity group and the confounded interactions with their alias sub-groups.

3. Converse Problem

Given a symmetrical fractional factorial design, the problems of finding out the identity contrasts, the confounded interactions, their alias sub groups and the factors that are to be suppressed in the analysis have been discussed in this section. We shall first solve this problem with reference to a particular case. The generalisation follows immediately from the example. Suppose the layout plan of $\frac{1}{2^3}(2^9, 2^4)$ design is given as in Appendix and we have to find out the identity group of interactions. In the layout the treatment combinations have been presented using alphabets to denote the non-zero levels of the factors for saving space. One can, however, convert such combination to one written by using 1 and 0 as levels of the factors. For subsequent discussion the treatment combinations have been written using 1 and 0 as levels.

At first we have to find out the three independent identity contrasts. For this six independent treatment combinations are chosen from the given fractional design such that with these treatments a (6×6) unit matrix can be formed by suitably choosing six of the nine columns corresponding to the 9 factors. The six independent treatment combinations, *viz., aghi, bghi, cghi, dg, eh, fi,* satisfy the above requirement and these form the following arrangement.

	Basic Factors						*Added Factors*		
	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Independent treatment combinations	1	0	0	0	0	0	1	1	1
	0	1	0	0	0	0	1	1	1
	0	0	1	0	0	0	1	1	1
	0	0	0	1	0	0	1	0	0
	0	0	0	0	1	0	0	1	0
	0	0	0	0	0	1	0	0	1

The six factors corresponding to these six columns form the basic factors, *viz., A, B, C, D, E, F* and the remaining three factors, *viz., G, H, I* are the added factors. The three columns under added factors give us the three independent identity contrasts, *viz., ABCDG, ABCEH* and *ABCFI* and these three added factors *G, H, I* are to be suppressed in the analysis. From these 3 independent contrasts we can generate 2^3-1 interactions that form the identity group of interactions.

Next, we have to find out the confounded interactions. For this, we have to choose 4 independent treatment combinations from the key block which is of size 2^4, such that a unit matrix (4×4) can be formed with four of the nine columns in these 4 treatment combinations. The 4 treatments are *aegi, bfgh, cghi, defghi.* From these 4 treatment combinations we have to choose 4 basic factors (columns), *viz.*, *A*, *B*, *C*, *D* which give a (4×4) unit matrix. The 4 treatments are arranged in the following way.

	Basic Factors				Added Factors				
	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Independent treatment combinations	1	0	0	0	1	0	1	0	1
	0	1	0	0	0	1	1	1	0
	0	0	1	0	0	0	1	1	1
	0	0	0	1	1	1	1	1	1

The remaining 5 factors (columns) *E*, *F*, *G*, *H*, & *I* form the added factors. The 5 columns under the added factors give the 5 independent confounded interactions, *viz.*, *ADE*, *BDF*, *ABCDG*, *BCDH*, *ACDI* which also include interactions belonging to the identity group. These 5 independent interactions generate 2^5-1 interactions that are confounded in the design. Now we can separate out the interactions confounded and the identity group of interactions by following exactly the same method described earlier while constructing such designs.

4. Generalised Case

The method of construction described in the case of fractional designs with factors each at two levels can easily be extended to fractional designs with factors having more than two levels. In the most general case, *i.e*, each factor has $s=p^n$ levels, where p is a prime number, if the element in the column under the added factors A_j be c_{ij} in the ith row, then the interaction confounded due to introduction of this column will consist of A_j^{p-1} togther with all r basic factors $A_i^{c_{ij}}$ $(i=1, 2, \ldots\ldots r)$. The rest of the method of construction remains the same.

5. To Obtain the Key Block of a Design Given a Set of Confounded Interactions, Through the Method of Das (1964)

Consider a $(2^6, 2^3)$ design and let the confounded interactions be *ACE, ADF, BCF, CDEF, ABEF, ABCD, BDE* From these interactions, we have to obtain the contents of the key block through the method of Das (1964). As the block size is 2^3, we should have 3 basic factors and 3 added factors. At first we have to find out the 3 added factors, from the given interactions. For this we have to choose first three independent interactions such that each of them contains a factor which does not occur in the other two. In this way we have to get three factors in the present case We find that in the interactions, *BCF, ADF,* and *ACE*, the factors *B, D* & *E* occur only once and hence they can be used as added factors. The three remaining factors, *viz.*, *A, C,* & *F* in this case are to be taken as basic factors. Under these basic factors, we have to form a (3×3) unit matrix. Then the three given interactions are to be written columnwise under the added factors. By this, we get the following arrangement.

	Basic Factors			*Added Factors*		
	A	*C*	*F*	*B*	*D*	*E*
Independent	1	0	0	0	1	1
treatment	0	1	0	1	0	1
combinations	0	0	1	1	1	0

Now factor-wise rearrangement so as to write the factors in the usual order, *viz* , *A, B, C, D, E,* & *F* can be done by suitably shifting the columns. Then we get the following arrangement.

	A	*B*	*C*	*D*	*E*	*F*
Independent	1	0	0	1	1	0
treatment	0	1	1	0	1	0
combinations	0	1	0	1	0	1

Row-wise we get the three independent treatment combinations and these generate 2^3 treatments along with the control which form the contents of the key block.

6. Summary

The construction of symmetrical fractional factorial designs with confounding, through the method of Das (1964) has been described. Through this method, the problem of selecting the factors that are to be suppressed in analysing the fractional factorial designs is automatically solved. Given a set of confounded interactions, obtaining the contents of the key block through the same method has also been described

Reference

Das, M.N. (1964): "A somewhat alternative approach for construction of symmetrical factorial designs and obtaining maximum number of factors." Calcutta Statistical Association Bulletin, Vol. 13, Nos. 49 and 50, pp. 1-17.

APPENDIX

$\frac{1}{2^3}(2^9, 2^4)$ Design.

Block 1	*Block 2*	*Block 3*	*Block 4*
(1)	dg	eh	fi
aegi	adei	aghi	aefg
bfgh	bdfh	befg	bghi
abefhi	abdefghi	abfi	abeh
cghi	cdhi	cegi	cfgh
aceh	acdegh	ac	acefhi
bcfi	bcdfgi	bcefhi	bc
abcefg	abcdef	abcfgh	abcegi
defghi	efhi	dfgi	degh
adfh	afgh	adef	adhi
bdei	begi	bdhi	bdef
abdg	ab	abdegh	abdfgi
cdef	cefg	cdfh	cdei
acdfgi	acfi	acdefghi	acdg
bcdegh	bceh	bcdg	bcdefghi
abcdhi	abcghi	abcdei	abcdfh